Schubert varieties in the Beilinson-Drinfeld Grassmannian

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$\mathfrak{g}$ — simple complex Lie algebra (e.g. $\mathfrak{sl}_n$).
$G \supset B \supset H$ — any corresponding Lie group, its Borel subgroup and maximal torus (e. g. $SL_n \supset$ {upper triangular matrices} $\supset$ {diagonal matrices}).
G/B — flag variety (e.g. variety of complete flags in $\mathbb{C}^n$).

Line bundles on the flag variety

For any torus character $\lambda : H \to \mathbb{C}^\times$ let $\mathbb{C}_\lambda$ be the corresponding 1-dimensional B-module, defined by: $B \twoheadrightarrow B/U = H \to \mathbb{C}^\times$ (U is the unipotent radical of B).
Define the line bundle $\mathcal{L}_\lambda$ on G/B by: $\mathcal{L}_\lambda = G \times_B \mathbb{C}_{-\lambda} \twoheadrightarrow G/B$.

Borel-Weil Theorem

For λ being a dominant weight there is an isomorphism of G-modules.

$$H^0(G/B, \mathcal{L}_\lambda) \simeq V(\lambda)^*,$$

where $V(\lambda)$ is the irreducible representation of the highest weight λ.

Schubert cells

The Bruhat decomposition:

$$G = \bigsqcup_{w \in W} BwB,$$

where W is the Weyl group. This implies:

$$G/B = \bigsqcup_{w \in W} Bw = \bigsqcup_{w \in W} X_w.$$

What is $H^0(\overline{X_w}, \mathcal{L}_\lambda)$?

Demazure modules

Let $w \in W$, $\lambda \in P^+$. There is a unique (up to scaling) vector $v_{w\lambda}$ of weight $w\lambda$ in $V(\lambda)$. Define the Demazure module:

$$V_w(\lambda) = U(\mathfrak{b}).v_{w\lambda} \subset V(\lambda).$$

Ex:

Demazure Theorem

$$H^0(\overline{X_w}, \mathcal{L}_\lambda) \simeq V_w(\lambda)^*.$$

Affine case

$G = G^{\mathrm{ad}}$ — the adjoint group of $\mathfrak{g}$ ($\pi_1(G) = P^\vee / Q^\vee$).
$\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K$ — affine Kac-Moody algebra.
$\mathrm{Gr}_G = G(\mathcal{K})/G(\mathcal{O})$ — affine Grassmannian, where $\mathcal{K} = \mathbb{C}((t)), \mathcal{O} = \mathbb{C}[[t]]$.
We have $\pi_0(\mathrm{Gr}_G) = \pi_1(G) = P^\vee / Q^\vee$, and:
$\mathrm{Gr}_G = \bigsqcup \mathrm{Gr}_G(\Lambda_i)$, where $\Lambda_0, \ldots, \Lambda_m$ — all level one integrable affine weights.
$\mathrm{Gr}_G(\Lambda_i) \subset \mathbb{P}(L(\Lambda_i))$.
There is a very ample determinant line bundle $\mathcal{L}$ on Gr_G.

Affine Borel-Weil theorem

$$H^0(\mathrm{Gr}_G, \mathcal{L}) \simeq \bigoplus_{i=1}^{m} L(\Lambda_i)^*.$$

More generally, for $\ell \geq 0$:

$$H^0(\mathrm{Gr}_G, \mathcal{L}^{\otimes \ell}) \simeq \bigoplus_{i=1}^{m} L(\ell\Lambda_i)^*.$$

Affine Schubert varieties and Demazure modules

Affine Schubert cells

$G(\mathcal{O})$ acts on Gr_G. In fact orbits are parametrized by dominant coweights:

$$\mathrm{Gr}_G = \bigsqcup_{\lambda^\vee \in P_+^\vee} G(\mathcal{O}).t^{\lambda^\vee} = \bigsqcup_{\lambda^\vee \in P_+^\vee} \mathrm{Gr}_G^{\lambda^\vee}.$$

What is $H^0(\overline{\mathrm{Gr}_G^{\lambda^\vee}}, \mathcal{L}^{\otimes \ell})$?

Affine Demazure modules

Example of $\mathfrak{sl}_2$:

For arbitrary $\mathfrak{g}$, there is an extremal vector v_λ of weight $\iota\lambda^\vee$ in one of $L(\Lambda_i)$, we define the affine Demazure module of level 1: $D(1, \lambda^\vee) = U(\mathfrak{g}[t]).v_\lambda \subset L(\Lambda_i)$.
For arbitrary level: in one of $L(\ell\Lambda_0), \dots, L(\ell\Lambda_m)$ there is an extremal vector of weight $\ell\iota\lambda^\vee$, we define the affine Demazure module by:

$$D(\ell, \lambda^\vee) = U(\mathfrak{g}[t]).v_\lambda.$$

Theorem (Kumar):

$$H^0(\overline{\mathrm{Gr}_G^{\lambda^\vee}}, \mathcal{L}^{\otimes \ell}) = D(\ell, \lambda^\vee)^*.$$

Moduli definition of the affine Grassmannian

Gr_G is the moduli space of pairs $(\mathcal{P};\beta)$, where $\mathcal{P}$ is a G-torsor on $\mathbb{A}^1$; $\beta : \mathcal{P}_{\mathbb{A}^1\setminus 0} \to G \times (\mathbb{A}^1 \setminus 0)$ is a trivialization on $\mathbb{A}^1 \setminus 0$.

Definition

The Beilinson-Drinfeld Grassmannian $\mathrm{Gr}_{\mathbb{A}^k}$ is the moduli space of collections consisting of the points $(c_1, \ldots, c_k) \in \mathbb{A}^k$, a G-torsor $\mathcal{P}$ over $\mathbb{A}^1$, and a trivialization of $\mathcal{P}$ outside points c_i.

Determinant line bundle

There is a determinant line bundle $\mathcal{L}$ on $\mathrm{Gr}_{\mathbb{A}^k}$ with the following property: for $\mathbf{c} = (c_1, \ldots, c_1, c_2, \ldots, c_2, \ldots, c_m, \ldots, c_m) \in \mathbb{A}^k$:

$$\mathcal{L}|_{\pi^{-1}(\mathbf{c})} = \mathcal{L}^{\boxtimes m} \text{ on } \mathrm{Gr}^{\times m}.$$

$\underline{\lambda}^\vee = (\lambda_1^\vee, \ldots, \lambda_k^\vee) \in (P^{\vee +})^k$. Let:

$$t^{\underline{\lambda}^\vee} : \mathbb{A}^k \to \mathrm{Gr}_{\mathbb{A}^k};$$

$$\text{for } \mathbf{c} \text{ s.t. } c_i \neq c_j, \mathbf{c} \mapsto (t^{\lambda_1^\vee}, \ldots, t^{\lambda_k^\vee}) \in \mathrm{Gr}^{\times k} = \pi^{-1}(\mathbf{c});$$

$$(c_1, \ldots, c_1, \ldots, c_m, \ldots, c_m) = \mathbf{c} \mapsto (t^{\lambda_1^\vee + \ldots + \lambda_{i_1}^\vee}, \ldots, t^{\lambda_{i_1 + \ldots + i_{m-1} + 1}^\vee + \ldots + \lambda_k^\vee}) \in \mathrm{Gr}^{\times m}.$$

Definition. Schubert cell

$\mathrm{Gr}_{\mathbb{A}^k}^{\underline{\lambda}^\vee} \subset \mathrm{Gr}_{\mathbb{A}^k}$, $\pi_{\underline{\lambda}^\vee} : \mathrm{Gr}_{\mathbb{A}^k}^{\underline{\lambda}^\vee} \to \mathbb{A}^k$, holds:

$$\pi_{\underline{\lambda}^\vee}^{-1}(\mathbf{c}) = G(\mathcal{O})^{\times m}.t^{\underline{\lambda}^\vee}(\mathbf{c}) \subset \mathrm{Gr}^{\times m};$$

$$\pi_{\underline{\lambda}^\vee}^{-1}(\mathbf{c}) = \mathrm{Gr}^{\lambda_1^\vee + \ldots + \lambda_{i_1}^\vee} \times \ldots \times \mathrm{Gr}^{\lambda_{i_1 + \ldots + i_{m-1} + 1}^\vee + \ldots + \lambda_k^\vee}.$$

Question

$$H^0(\overline{\mathrm{Gr}}^{\lambda^\vee}_{\mathbb{A}^k}, \mathcal{L}^{\otimes \ell}) = ?$$

Answer

Theorem (Feigin, Finkelberg, D.).

$$H^0(\overline{\mathrm{Gr}}^{\underline{\lambda}^\vee}_{\mathbb{A}^k}, \mathcal{L}^{\otimes \ell})^\vee \simeq \mathbb{D}(\ell, \underline{\lambda}^\vee) \otimes_{\mathcal{A}(\lambda)} \mathbb{C}[\mathbb{A}^k].$$

Globalization of a single Demazure module

We define $\mathbb{D}(\ell, \lambda^\vee) = D(\ell, \lambda^\vee)[t]$ with $\mathfrak{g}[t]$ action defined by

$$xt^m.(v \otimes t^k) = \sum_{j=0}^{m} \binom{m}{j} (xt^j v) \otimes t^{m+k-j}.$$

Global Demazure module

Suppose $\underline{\lambda}^\vee = (\lambda_1^\vee, \ldots, \lambda_k^\vee)$. Then define:

$$\mathbb{D}(\ell, \underline{\lambda}^\vee) = U(\mathfrak{g}[t]).(v_1 \otimes \ldots \otimes v_k) \subset \mathbb{D}(\ell, \lambda_1^\vee) \otimes \ldots \otimes \mathbb{D}(\ell, \lambda_k^\vee).$$

Reason

Global Demazure module is a natural higher-level generalization of Global Weyl module: for simply-laced $\mathfrak{g}$ and fundamental λ_i holds an isomorphism $\mathbb{D}(1, \underline{\lambda}) \simeq \mathbb{W}_{\lambda_1 + \ldots + \lambda_k}$.

$\mathbb{D}(\ell, \underline{\lambda}^\vee)$ admits an action of $\mathfrak{h}[t]$, which commutes with the natural $\mathfrak{g}[t]$-action. For the cyclic vector $v_{\underline{\lambda}} \in \mathbb{D}(\ell, \underline{\lambda}^\vee)$:

$$ht^m.v_{\underline{\lambda}} = ht^m v_{\underline{\lambda}},$$

and for any $u \in U(\mathfrak{g}[t])$:

$$ht^m.(uv_{\underline{\lambda}}) = u(ht^k)v_{\underline{\lambda}}.$$

Define the highest weight algebra $\mathcal{A}(\underline{\lambda}) = U(\mathfrak{h}[t])/\mathrm{Ann}\, v_{\underline{\lambda}}$. We naturally have:

$$\mathcal{A}(\underline{\lambda}) = \mathrm{hw}\mathbb{D}(\ell, \underline{\lambda}^\vee) \subset \mathrm{hw}(\bigotimes_{i=1}^{k} D(\ell, \lambda_i^\vee)[t]) = \bigotimes_{i=1}^{k} \mathbb{C}[t] = \mathbb{C}[\mathbb{A}^k].$$

Theorem (Feigin, Finkelberg, D.)

$$H^0(\overline{\mathrm{Gr}}^{\underline{\lambda}^\vee}_{\mathbb{A}^k}, \mathcal{L}^{\otimes \ell})^\vee \simeq \mathbb{D}(\ell, \underline{\lambda}^\vee) \otimes_{\mathcal{A}(\lambda)} \mathbb{C}[\mathbb{A}^k],$$

where $M^\vee$ stands for the $\mathbb{C}[\mathbb{A}^k]$-dual module.

Free Azat!

Glory to Ukraine!